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Amalgamation in the pentagon variety

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Abstract. The amalgamation class Amal $(\mathcal{N})$ of a lattice variety generated by a pentagon is considered. It is shown that Amal $(\mathcal{N})$ is closed under reduced products and therefore is an elementary class determined by Horn sentences. The above result is based on a new characterization of Amal $(\mathcal{N})$. The lattice varieties whose amalgamation classes contain Amal $(\mathcal{N})$ as a subclass are considered.

Introduction

We consider the amalgamation class Amal $(\mathcal{N})$ of a lattice variety generated by the pentagon. The central result of the paper is Theorem 3.1, that Amal $(\mathcal{N})$ is an elementary class. It is closed under reduced products and therefore is determined by Horn sentences. Furthermore, if B is a subdirect power of the pentagon and also an image of $A \in$ Amal $(\mathcal{N})$, then $B \in$ Amal $(\mathcal{N})$.

This result contrasts with the following of Bergman (see [2] Theorem 2): If $\mathcal{V}$ is a variety generated by a finite modular lattice then Amal $(\mathcal{V})$ is not an elementary class.

The proof of Theorem 3.1 is based on Theorem 2.2 which gives a new characterization of membership of Amal $(\mathcal{N})$. The first characterization was given by Bergman (see [9] Theorem 6.2).

The examples of lattice varieties whose amalgamation class contains Amal $(\mathcal{N})$ as a subclass are considered in Proposition 4.1.

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0. Preliminaries

0.1. *Notation, terminology and some required results*

0.1.1. Let u/v be a quotient of a lattice L. We shall say that u/v is an *N-quotient* of L, if for some $z \in L$ the set $\{u, v, z\}$ generates a sublattice of L isomorphic to a pentagon N (see Figure 1) with u/v as a critical quotient of this pentagon. In this case we write $N(u/v, z)$. (Note that c/a is a critical quotient of N in Figure 1.)

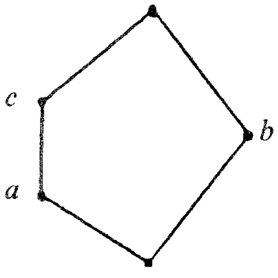
0.1.2. We shall use $\underline{1}$, $\underline{2}$, and $\underline{3}$ to denote one-, two- and three-element chains respectively. If $X \cong \underline{2}$ ($X \cong \underline{3}$) and X is a sublattice of a lattice L we say that X is a $\underline{2}$- ($\underline{3}$-) sublattice of L if there is a retraction of L onto X.

COROLLARY. *Let $\{p > q\} \leq N$ and assume that $\{p, q\} \neq \{c, a\}$. If K is any sublattice of N which contains $\{p, q\}$, then $\{p, q\}$ is a $\underline{2}$-sublattice of K.*

0.1.3. Con (A) denotes the *congruence lattice* of an algebra A. The smallest congruence on A which identifies $a, b \in A$ is denoted by Con (a, b).

0.1.4. Let $f: A \to B$ be an epimorphism. The kernel of f, $\ker f$, is called a *B-congruence* on A.

0.1.5. Let $A = \prod_{i \in I} A_i$ be a product of algebras. The congruence φ on A is said to be *filtral* if it is induced by an ultrafilter on I. It is well-known that if $\{A_i : i \in I\}$ is a finite set of finite algebras then every filtral congruence is an A_i-congruence for some $i \in I$.



N

Figure 1

Let A be as in the paragraph above and assume that A is congruence distributive. We shall frequently use the following fundamental result:

JÓNSSON'S LEMMA. *If B is a subalgebra of A and $\theta \in \operatorname{Con}(B)$ is completely meet irreducible then there is a filtral congruence φ on A such that $\varphi|_B \subset \theta$.*

0.1.6. Let A be as in 0.1.5. A subalgebra B of A is *regular* if for any two kernels θ_i, θ_j of distinct projections on A we have $\theta_i|_B \neq \theta_j|_B$.

0.1.7. The variety generated by N is denoted $\mathcal{N}$. The last two statements of this subsection are concerned with members of $\mathcal{N}$.

0.1.8. Let ψ be a congruence on a lattice L. Call ψ distributive if L/ψ is a distributive lattice.

COROLLARY. *Let $L \in \mathcal{N}$. Then $\psi \in \operatorname{Con}(L)$ is the smallest distributive congruence on L if and only if*

$$\psi = \bigcap \theta_i = \sum \operatorname{Con}(u_s, v_s)$$

where $\{\theta_i : i \in I\}$ is the set of all $\underline{2}$-congruences on L and $\{u_s/v_s : s \in S\}$ is the set of all N-quotients of L (certainly if L is distributive S is empty).

Proof. The equality $\psi = \bigcap \theta_i$ holds trivially since every distributive lattice is a subdirect product of two-element chains. To prove the equality $\psi = \sum \operatorname{Con}(u_s, v_s)$, we first observe that the inclusion $\sum \operatorname{Con}(u_s, v_s) \subset \psi$ holds in $\operatorname{Con}(L)$ for any lattice L. Furthermore this inclusion must be an equality if $L \in \mathcal{N}$ since in that case L does not have a diamond as its sublattice.

0.1.9. Let A be a subdirect power of N and let p/q be a non-trivial quotient of A. Then for some N-congruence φ on A we have $(p, q) \notin \varphi$ so that $(p/\varphi)/(q/\varphi)$ is a non-trivial quotient of a pentagon A/φ. Since the pentagon is a projective lattice, there is an N-quotient u/v of A such that $(u/\varphi)/(v/\varphi)$ is a critical quotient of the pentagon A/φ. Since $(u/\varphi)/(v/\varphi)$ projects weakly onto $(p/\varphi)/(v/\varphi)$ in A/φ it follows that there is a quotient u'/v' of A which projects weakly onto both u/v and p/q in at most five steps (see [11]). Since $A \in \mathcal{N}$ we have that u'/v' is an N-quotient (see [10]). Thus A is a subdirect power of N if and only if for any non-trivial quotient p/q of A there is an N-quotient u/v of A which projects weakly onto p/q in at most five steps.

Although the following result is known we shall prove it to make the paper self-contained.

LEMMA. *Let B be a reduced product of the family $\{A_i : i \in I\}$ and assume that each A_i is a subdirect power of N. Then B is a subdirect power of N.*

Proof. Let $A = \prod A_i$ and $\psi \in \mathrm{Con}(A)$ such that $A/\psi = B$. Let $\mathscr{D}$ be a filter on I which induces ψ. Let $(p/\psi)/(q/\psi)$ be a non-trivial quotient of $B(p, q \in A)$. Then

$$X = \{i \in I : p_i > q_i\} \in \mathscr{D}.$$

Since each A_i is a subdirect power of N for any $i \in X$ there is an N-quotient $u_i v_i$ of A_i such that u_i/v_i projects weakly onto p_i/q_i in at most five steps. Consider a quotient u/v of A which is described as follows:

If $i \in X$ then u_i/v_i is an N-quotient which projects onto p_i/q_i.

If $i \notin X$ then $u_i = v_i$.

Then u/v is an N-quotient which projects weakly on p/q in A so that $(u/\psi)/(v/\psi)$ is an N-quotient in B which projects weakly onto $(p/\psi)/(q/\psi)$.

0.2. *Absolute retracts and amalgamation in some congruence distributive varieties*

By a *diagram* in a variety $\mathscr{V}$ we mean a quintuple (A, f, B, g, C) with $A, B, C \in \mathscr{V}$ and embeddings $f : A \to B$, $g : A \to C$. By an *amalgam* in $\mathscr{V}$ of this diagram we mean a triple (D, f_1, g_1) with $D \in \mathscr{V}$ and $f_1 : B \to D$, $g_1 : C \to D$ embeddings such that $ff_1 = gg_1$. If such an amalgam exists we say that the diagram can be amalgamated in $\mathscr{V}$. An algebra $A \in \mathscr{V}$ is called an amalgamation base for $\mathscr{V}$ if every diagram (A, f, B, g, C) can be amalgamated in $\mathscr{V}$. The class of all amalgamation bases for $\mathscr{V}$ is called the *amalgamation class* and is denoted $\mathrm{Amal}(\mathscr{V})$.

It is known that $\mathrm{Amal}(\mathscr{V})$ is a proper class ([12]).

An algebra $A \in \mathscr{V}$ is said to be an *absolute retract* in $\mathscr{V}$ if for any embedding $f : A \to B \in \mathscr{V}$ there is an epimorphism $g : B \to A$ such that gf is an identity map on A. Absolute retracts of a variety $\mathscr{V}$ belong to $\mathrm{Amal}(\mathscr{V})$ (see [1] Proposition 3.2).

EXAMPLE 0.2.1. Let $\mathscr{V}$ be a residually small, congruence distributive variety. Then the collection of subdirectly irreducible members of $\mathscr{V}$ forms a set (up to equivalence). The subdirectly irreducible members which have no essential extensions are absolute retracts in $\mathscr{V}$ (see [1]). We call them *maximal subdirectly irreducible* members of $\mathscr{V}$. The set of maximal subdirectly irreducible members of $\mathscr{V}$ is denoted by $\mathscr{V}_{MI}$.

For instance if $\mathcal{N}$ is the variety generated by a pentagon then $\mathcal{N}_{MI} = \{N\}$. Similarly if L is any lattice on Figures 2 and 3 and $\mathcal{L}$ is the variety generated by L then $\mathcal{L}_{MI} = \{L\}$.

The proof of the following result may be found in [3] and [7].

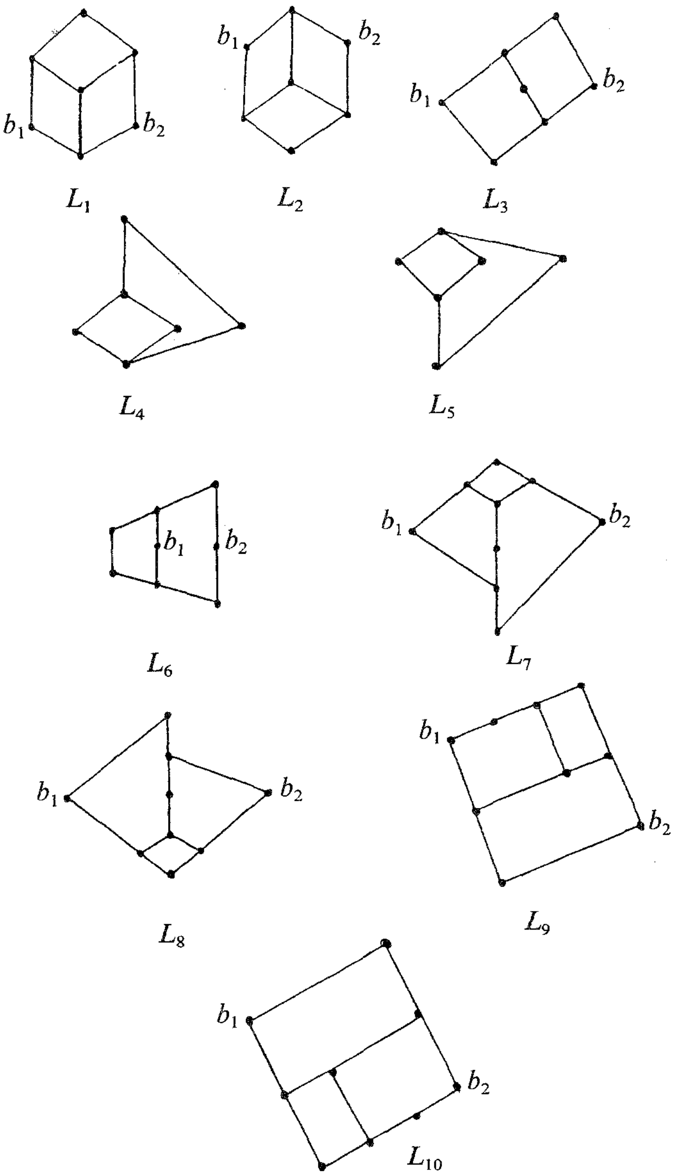
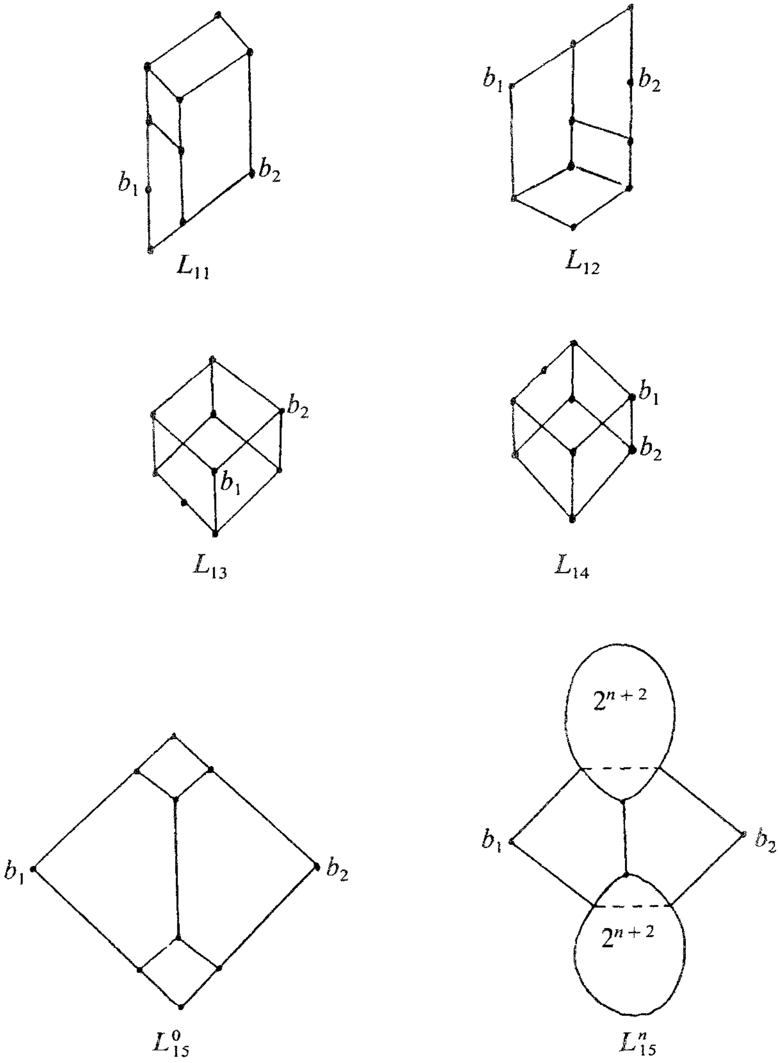


Figure 2



L_{11} L_{12} L_{13} L_{14} L_{15}^0 L_{15}^n

Figure 3

THEOREM 0.2.2. *Let $\mathscr{V}$ be a residually small, congruence distributive variety and assume that every member of $\mathscr{V}$ has a one-element subalgebra. Then*

(1) Amal $(\mathscr{V})$ *is closed under finite products.*

(2) $A \in$ Amal $(\mathscr{V})$ *iff for any embedding $f : A \to B \in \mathscr{V}$ and any homomorphism $g : A \to M \in \mathscr{V}_{MI}$ there is a homomorphism $h : B \to M$ such that $g = hf$.*

(3) *A product of absolute retracts of $\mathscr{V}$ is an absolute retract and therefore belongs to* Amal $(\mathscr{V})$.

EXAMPLE 0.2.3. (1) Let L be one of the lattices of Figures 1, 2 or 3 and let $\mathcal{V}$ be the variety generated by L. Then for the arbitrary index set I the direct power L^I is an absolute retract in $\mathcal{V}$ and therefore belongs to Amal ($\mathcal{V}$).

(2) We show later (Proposition 4.1) that if $L = L_1$, L_2 or $L = L_{15}^n$ for some $n \in \omega$ then Amal ($\mathcal{V}$) $\subset$ Amal ($\mathcal{V}$). On the other hand, if $L \in \{L_3, \dots, L_{14}\}$, then Amal ($\mathcal{N}$) is not contained in Amal ($\mathcal{V}$). This can easily be seen from Theorem 0.2.2(2): consider the embeddings f and g of N into L such that $f(b) = b_1$ and $g(b) = b_2$. Then there is no $h : L \to L$ such that $g = hf$.

The proof of the following lemma can be found in [6].

LEMMA 0.2.4. *Let $\mathcal{V}$ be an arbitrary variety. A diagram (A, f_0, B_0, f_1, B_1) can be amalgamated in $\mathcal{V}$ iff for $\{i, j\} = \{0, 1\}$ and $a \neq b \in B_i$ there is an algebra $D \in \mathcal{V}$ and homomorphisms $g : B_i \to D$ and $h : B_j \to D$ such that $gf_i = hf_j$ and $g(a) \neq g(b)$.*

1. Congruences and homomorphisms

The results of this section are mainly technical. They are needed to prove Theorems 2.2 and 3.1.

1.1. *Finitely generated congruence distributive varieties*

LEMMA 1.1.1. *Let $\mathcal{V}$ be a congruence distributive variety generated by a finite set $\{M_1, \dots, M_n\}$ of its finite maximal subdirectly irreducible members. Suppose further that, for distinct $i, j \in \{1, \dots, n\}$, M_i is not an image of a subalgebra of M_j. Then if $A \leq B \in \mathcal{V}$, every M_i-congruence on A can be extended to an M_i-congruence on B.*

Proof. Since we have $B \leq M_{s_1}^{I_1} \times \cdots \times M_{s_k}^{I_k}$ where $s_1, \dots, s_k \in \{1, \dots, n\}$, it suffices to prove the lemma under the assumption that $B = M_{s_1}^{I_1} \times \cdots \times M_{s_k}^{I_k}$. Let θ be an M_i-congruence on A. By Jónsson's Lemma there is a filtral congruence $\psi \in \operatorname{Con}(B)$ such that $\psi|_A \subset \theta$. Since the set $\{s_1, \dots, s_k\}$ is finite and each $M_{s_1}, \dots, M_{s_k}$ is also finite, we have $B/\psi \cong M_{s_t}$ for some $t \in \{1, \dots, k\}$. It follows from the second assumption of the lemma that $\psi|_A = \theta$ so that $M_{s_t} = M_i$.

COROLLARY 1.1.2. *Let $\mathcal{V}$ be as in Lemma* 1.1.1 *and let B and C be products of members of $\{M_1, \dots, M_n\}$. If $f : A \to B$ and $g : A \to C$ are subdirect representations then the diagram (A, f, B, g, C) can be amalgamated in $\mathcal{V}$.*

Proof. We use Lemma 0.2.4. Let $u \neq v \in B$ so that $\pi(u) \neq \pi(v)$ for some projection π on B. Since f is a subdirect representation the map πf is an epimorphism of

A onto some member of $\{M_1, \ldots, M_n\}$. It follows from Lemma 1.1.1 that there is a map h on C such that $hg = \pi f$.

COROLLARY 1.1.3. *Let* $\mathscr{V}$ *be as in Lemma* 1.1.1 *and assume that* A *is a subdirect product of members of* $\mathscr{V}_{MI}$. *Then* $A \in \mathrm{Amal}(\mathscr{V})$ *iff for any regular subdirect representation* $f: A \to B$ *and any homomorphism* $k: A \to M \in \mathscr{V}_{MI}$ *there is a homomorphism* $h: B \to M$ *such that* $k = hf$.

Proof. The necessity follows from Theorem 0.2.2(2). To prove that the conditions are sufficient we have to show that if B, C are products of members of $\mathscr{V}_{MI}$ and $f: A \to B$, $g: A \to C$ are embeddings, then the diagram (A, f, B, g, C) can be amalgamated in $\mathscr{V}$. (This is because every member of $\mathscr{V}$ is embeddable into a product of members of $\mathscr{V}_{MI}$.)

Now since A is a subdirect product of members of $\mathscr{V}_{MI}$, the embeddings f and g induce regular subdirect representations $f': A \to B' \leq B$ and $g': A \to C' \leq B$ (see Figure 4). By Corollary 1.1.2 the diagram (A, f', B', g', C') has an amalgam H. We may assume $H \in \mathrm{Amal}(\mathscr{V})$. To complete the proof it is sufficient to show that the diagrams (A, f, B, f', B') and (A, g, C, g', C') have amalgams (D, s_1, t) and (E, l_1, m). Indeed, if such amalgams exist then, since $B', C' \in \mathrm{Amal}(\mathscr{V})$ by Theorem 0.2.2(3)), the diagrams (B', q, H, t, D) and (C, m, E, p, H) will have amalgams (F, v, s_2) and (G, u, l_2).

Further, since $H \in \mathrm{Amal}(\mathscr{V})$, the diagram (H, v, F, u, G) will have an amalgam (X, s_3, t_3) so that $(X, s_3 s_2 s_1, l_3 l_2 l_1)$ is an amalgam of (A, f, B, g, C).

So by symmetry we have to show that the diagram (A, f, B, f', B') has an amalgam (D, s_1, t). We use Lemma 0.2.4. We have $B = M_1^{I_1} \times \cdots \times M_n^{I_n}$ and $B' = M_{z_1}^{J_{z_1}} \times \cdots \times M_{z_y}^{J_{z_y}}$ where $y \leq n$ and $J_{z_1} \subset I_{z_1}, \ldots, J_{z_y} \subset I_{z_y}$. Now let $z \neq w \in B$ so that $\pi(z) \neq \pi(w)$ for some projection π on B. Then πf is a homomorphism of A into some member of $\mathscr{V}_{MI}$ so that by the assumption of the corollary there is a homomorphism h on B' into the same member of $\mathscr{V}_{MI}$ such that $hf' = \pi f$.

Finally let $z \neq w \in B'$ and let π be a projection on B' such that $\pi(z) \neq \pi(w)$. Since f' is a subdirect representation, $\pi f'$ is an epimorphism of A onto some member of $\mathscr{V}_{MI}$. It follows from Lemma 1.1.1 that there is an epimorphism h on B onto the same member of $\mathscr{V}_{MI}$ such that $hf = \pi f'$. This completes the proof.

1.2. *On extension of* $\underline{2}$*-congruences of members of* $\mathscr{N}$

We begin with the following simple result:

LEMMA 1.2.1. *Let* $f: A \to B \in \mathscr{N}$ *be an embedding and assume that every* $\underline{2}$*-congruence on* $f(A)$ *can be extended to a* $\underline{2}$*-congruence on* B. *Then if* C *is a finite*

$$\begin{tikzcd}
 & & & X & & & \\
 & & G \arrow[ur, "l_3"] & & F \arrow[ul, "s_3"'] & & \\
 & & & H \arrow[ul, "u"'] \arrow[ur, "v"] & & & \\
 & E \arrow[uur, "l_2"] & & & & D \arrow[uul, "s_2"'] & \\
C \arrow[ur, "l_1"] & & C' \arrow[ul, "m"'] \arrow[uur, "p"] & & B' \arrow[uul, "q"'] \arrow[ur, "t"] & & B \arrow[ul, "s_1"'] \\
 & & & A \arrow[ulll, "g"] \arrow[ul, "g'"'] \arrow[ur, "f'"] \arrow[urrr, "f"'] & & &
\end{tikzcd}$$

Figure 4

Boolean algebra and $g : A \to C$ *is an epimorphism, there is an epimorphism* $h : B \to C$ *such that* $g = hf$.

One of the main properties of members of Amal $(\mathcal{N})$ is described in the following:

LEMMA 1.2.2. *Let $A \in \operatorname{Amal}(\mathcal{N})$ and $A \leq B \in \mathcal{N}$. Then every $\underline{2}$-congruence on A can be extended to a $\underline{2}$-congruence on B.*

Proof. Let $g : A \to N$ be a homomorphism with $g(A) \cong \underline{2}$ and $g(A) \neq \{a, c\}$ (see Figure 1). By Theorem 0.2.2(2) there is $h : B \to N$ such that $h(A) = g(A)$. By Corollary 0.1.2 there is a retraction k from $h(B)$ onto $g(A)$ so that the kernel of kh is a $\underline{2}$-congruence on B which extends $\ker g$.

LEMMA 1.2.3. *Let $A \leq B \in \mathcal{N}$. Then the following are equivalent:*

(1) *Every $\underline{2}$-congruence on A can be extended to a $\underline{2}$-congruence on B.*

(2) *If θ is a distributive congruence on A, u/v is an N-quotient of B and* $\operatorname{Con}(u, v) \in \operatorname{Con}(B)$ *then* $\operatorname{Con}(u, v)|_A \subset \theta$.

Proof. Let ψ_A and ψ_B be the smallest distributive congruences on A and B respectively. Consider the following sets:

$X_B =$ the set of all $\underline{2}$-congruences on B

$X_A = \{\varphi \in X_B : \varphi|_A \text{ is a } \underline{2}\text{-congruence on } A\}$

$T_B =$ the set of all N-quotients of B.

(1) implies (2): Assume (1). Then $\bigcap X_A = \bar{\psi}_A \in \operatorname{Con}(B)$ is an extension of ψ_A. Since $X_A \subset X_B$ we have

$$\bar{\psi}_A = \bigcap X_A \supset \bigcap X_B = \psi_B.$$

On the other hand, since $\psi_B|_A$ is a distributive congruence on B, we always have $\psi_A \subset \psi_B|_A$. Thus it follows from (1) that

$$\psi_A = \bar{\psi}_A|_A = \psi_B|_A.$$

Since $\psi_B \supset \operatorname{Con}(u, v)$ for any $u/v \in T_B$ (see Corollary 0.1.8), (1) implies (2). (2) implies (1): Let θ be a $\underline{2}$-congruence on A. Since every member of $\mathcal{N}$ is embeddable into a direct power of N, without loss of generality we may assume that $B = N^I$. By Jónsson's Lemma there is a filtral congruence φ on B such that $\varphi|_A \subset \theta$. Let $u/v \in T_B$ such that $u/\varphi/_{v/\varphi}$ is a critical quotient of a pentagon B/φ and let $\{x, y\}$ be a $\underline{2}$-sublattice of A such that $A/\theta = \{x/\theta, y/\theta\}$. Since (2) holds we have $\operatorname{Con}(u, v)|_A \subset \theta$ and therefore $\{x/\varphi, y/\varphi\}$ cannot be a critical quotient of the pentagon B/φ. It follows from Corollary 0.1.2 that there is a retraction h of B/φ onto $\{x/\varphi, y/\varphi\}$. Furthermore h can be chosen so that, for $p \in A$ and $q \in \{x, y\}$,

$h(p/\varphi) = h(q/\varphi)$ iff $p\theta q$. Let $k : B \to B/\varphi$ be a canonical epimorphism. Then ker (hk) is a $\underline{2}$-congruence on B which extends θ.

COROLLARY 1.2.4. *Let N^I be a direct power of the pentagon and let $f : A \to N^I$ be an embedding. Assume that every $\underline{2}$-congruence on $f(A)$ can be extended to a $\underline{2}$-congruence on N^I. If $g : A \to N$ is a homomorphism such that $g(A) \cong \underline{3}$ and the critical quotient of N is a sublattice of $g(A)$ then there is no $h : N^I \to N$ such that $g = hf$.*

Proof. It follows from Lemma 1.2.3 that if such h existed, then $h(N^I)$ would be a distributive sublattice of N. However, since the critical quotient of N is a sublattice of $g(A)$, the lattice $h(N^I)$ must be a chain having at least three elements. This is impossible since every distributive image of N^I is a Boolean algebra (see also [2] Theorem 7).

Our next result is concerned with restrictions of $\underline{2}$-congruences of a product of members of $\mathcal{N}$ into its factors.

LEMMA 1.2.5. *Let $A = \prod_{\gamma \in \alpha} A_\gamma$ be a product of members of $\mathcal{N}$. If θ is a $\underline{2}$-congruence on A then for some $\gamma \in \alpha$ there is an embedding $A_\gamma \leq A$ such that $\theta|_{A_\gamma}$ is a 2-congruence on A_γ.*

Proof. Let $\{u > v\} \subset A$ such that $A/\theta = \{u/\theta, v/\theta\}$ and let

$$\Omega = \{\beta \in \alpha : u_\beta > v_\beta\}.$$

(Here u_β and v_β are the βth coordinates of u and v respectively.) Consider two possible cases:

Case 1. For some $\gamma \in \Omega$ the set $\{u_\gamma > v_\gamma\} \subset A_\gamma$ is a $\underline{2}$-sublattice of A_γ. Let $f : A_{\gamma\gamma} \to \underline{2} = \{0, 1\}$ be an epimorphism with $f(u_\gamma) = 1$ and $f(v_\gamma) = 0$. Consider an embedding $A_\gamma \leq A$ given by

$$x \to \bar{x} \qquad (x \in A_\gamma, \bar{x} \in A)$$

which can be described as follows:

(1) $\bar{x}_\gamma = x$

(2) If $\beta \in \Omega$ and $\beta \neq \gamma$ then

$$\bar{x}_\beta = u_\beta \quad \text{if} \quad f(x) = 1 \text{ and}$$
$$\bar{x}_\beta = v_\beta \quad \text{if} \quad f(x) = 0.$$

(3) If $\beta \in (\alpha - \Omega)$ then $\bar{x}_\beta = u_\beta = v_\beta$.

It follows from (1), (2) and (3) that $\{u > v\} \subset A_\gamma \leq A$ so that $\theta|_{A_\gamma}$ is a $\underset{\sim}{2}$-congruence on A_γ.

Case 2. Not Case 1, i.e. for any $\beta \in \Omega$ the set $\{u_\beta > v_\beta\} \subset A_\beta$ is not a $\underset{\sim}{2}$-sublattice of A_β.

Since θ is completely meet irreducible in Con (A), by Jónsson's Lemma there is a filtral congruence ψ on A such that $\psi \subset \theta$. Let $\mathscr{D}$ be an ultrafilter on α which induces ψ. We have $\Omega \in \mathscr{D}$. Furthermore it follows from the assumption which is made in Case 2 that, for any $\beta \in \Omega$, the set $\{\beta\}$ does not belong to $\mathscr{D}$. Thus $(\Omega - \{\beta\}) \in \mathscr{D}$ for $\beta \in \Omega$. Choose any $\gamma \in \Omega$. Since $\{u_\gamma > v_\gamma\} \subset A_\gamma$, the lattice A_γ is non-trivial and, since $A_\gamma \in \mathscr{N}$, there is at least one epimorphism $f: A_\gamma \to \underset{\sim}{2} = \{0, 1\}$. Consider an embedding $A_\gamma \leq A$ associated with f which is similar to that of Case 1. Let $\{x > y\} \subset A_\gamma$ be such that $f(x) = 1$ and $f(y) = 0$. It follows that $u/\theta = \bar{x}/\theta$ and $v/\theta = \bar{y}/\theta$ so that $\theta|_{A_\gamma}$ is a $\underset{\sim}{2}$-congruence on A_γ.

For our next result we assume that B is a subdirect power of N and B is an image of $A \in \text{Amal}(\mathscr{N})$. Let $\Omega \in \text{Con}(A)$ be such that $A/\Omega \cong B$. for $a \in A$ we let $\bar{a} \in B$ be the image of a/Ω under the isomorphism between A/Ω and B. Also if $\theta \in \text{Con}(A)$ with $\Omega \leq \theta$ we let $\bar{\theta} \in \text{Con}(B)$ be the image of θ under the isomorphism between the principal filter in Con (A) generated by Ω and the lattice Con (B).

PROPOSITION 1.2.6. *Let $B \leq N^I$ be a regular subdirect representation. If $\bar{\theta}$ is a $\underset{\sim}{2}$-congruence on B then $\bar{\theta}$ can be extended to a $\underset{\sim}{2}$-congruence on N^I.*

Proof. See below.

Let $\{\varphi_s : s \in S\}$ be a set of all N-congruences on A, so that each φ_s is a kernel of an epimorphism of A onto N. Since the subdirect representation $B \leq N^I$ in Proposition 1.2.6 is regular we may assume that $I \subset S$. Let $J = S - I$ so that $N^S = N^I \times N^J$.

By Theorem 2.1(1) below A is a subdirect power of N so we may consider a subdirect representation $A \leq N^S$ such that for $a \in A$:

$$a_i = \bar{a}_i \quad \text{if } i \in I$$

$$a_j = (a/\varphi_j) \in N \quad \text{if } j \in J.$$

Let $\bar{\theta} \in \text{Con}(B)$ as in Proposition 1.2.6 and consider $\{\bar{u} > \bar{v}\} \subset B$ such that $B/\bar{\theta} = \{\bar{u}/\bar{\theta}, \bar{v}/\bar{\theta}\}$. Then for $\{u > v\} \subset A$ we have $A/\theta = \{u/\theta, v/\theta\}$. Since

$A \in \mathrm{Amal}(\mathcal{N})$, by Lemma 1.2.2 θ can be extended to a $\underline{2}$-congruence Σ on N^S. By Jónsson's Lemma there is a filtral congruence ψ on N^S such that $\psi \subset \Sigma$. Let $\mathcal{D}$ be an ultrafilter on S which induces ψ. Since $N^S/\psi \cong N$ and $N^S/\Sigma = \{u/\Sigma, v/\Sigma\}$ we have that for some $\{p, q\} \neq \{a, c\}$ (see Figure 1) the set $X = \{s \in S : u_s = p > q = v_s\}$ belongs to $\mathcal{D}$. Now if $(X \cap I) \in \mathcal{D}$ then Proposition 1.2.6 holds at once since, for $i \in I$, u and v have the same ith coordinates as $\bar{u}$ and $\bar{v}$. So assume that $(X \cap J) = Y \in \mathcal{D}$. In order to prove Proposition 1.2.6 under this assumption we need a lemma.

LEMMA 1.2.7. (1) *There is an embedding* $f: B \to N^S$ *such that* $\Sigma|_{f(B)} = \bar{\theta}$.

(2) *There is an embedding* $g: N^I \to N^S$ *such that* $\Sigma|_{g(N^I)} = \Delta$ *is a* $\underline{2}$*-congruence on* N^I.

(3) *Let* $h: N^I \to N^S$ *be an embedding such that for* $y, z \in N^I$

$$[h(y)]_i = y_i \qquad \text{for } i \in I$$

$$[h(y)]_j = [h(z)]_j \qquad \text{for } j \in J.$$

Then there is a $\underline{2}$*-congruence* Σ' *on* N^S *such that* $\Sigma'|_{h(N^I)} = \Delta$ *where* Δ *is as in* (2).

Proof. (1) Let $t : B \to \underline{2} = \{0, 1\}$ be an epimorphism with $\ker t = \bar{\theta}$. Then f can be described as follows:

$$[f(\bar{a})]_i = \bar{a} = a_i \qquad \text{for } \bar{a} \in B \text{ and } i \in I.$$

For $y \in Y$,

$$[f(\bar{a})]_y = p = u_y \qquad \text{if } t(\bar{a}) = 1$$

and

$$[f(\bar{a})]_y = q = v_y \qquad \text{if } t(\bar{a}) = 0.$$

For $s \in S - (I \cup Y)$ and $\bar{a}, \bar{b} \in B$ $[f(\bar{a})]_s = [f(\bar{b})]_s$.

(2) The idea here is similar to the one in proof of (1). Consider any epimorphism $t : N^I \to \underline{2} = \{0, 1\}$. Then g is described as follows:

$$[g(k)]_i = k_i \qquad \text{for } k \in N^I \text{ and } i \in I.$$

For $y \in Y$,

$$[g(k)]_y = p = u_y \qquad \text{if } t(k) = 1$$

and

$$[g(k)]_y = q = v_y \qquad \text{if } t(k) = 0.$$

For $s \in S - (I \cup Y)$ and $k, l \in N^I$, $[g(k)]_s = [g(l)]_s$.

(3) Let $h : N^I \to N^S$ be as in (3) and $g : N^I \to N^S$ be as in (2). Since $N^I \in \mathrm{Amal}(\mathcal{N})$ (see Theorem 0.2.2(3) and Example 0.2.1), the diagram (N^I, g, h, N^S) has an amalgam (Q, g_0, h_0) with $Q \in \mathcal{N}$ (see Figure 5). Since $N^S \in \mathrm{Amal}(\mathcal{N})$, by Lemma 1.2.2, Σ can be extended to a $\underline{2}$-congruence Π on Q. Letting $\Sigma' = \Pi|_{h_0(N^S)}$, we have $\Sigma'|_{h_0 h(N^I)} = \Delta$.

Proof of Proposition 1.2.6. As we noticed before we only need to consider the case $(X \cap J) = Y \in \mathcal{D}$. Let f and h be as in Lemma 1.2.7(1) and (3). Since $B \leq N^I$ and $h : N^I \to N^S$ we can consider a diagram (B, f, h, N^S). Since the variety of all lattices satisfies the amalgamation property (see [8]), the above diagram has an amalgam of (R, f_0, h_0) where R is a lattice not necessarily in $\mathcal{N}$ (see Figure 6). The proposition follows now from Lemma 1.2.7 since $\bar{\theta} = \Sigma|_{f_0 f(B)} = \Sigma'|_{h h_0(B)} = \Delta|_{h_0 h(B)}$.

2. Members of Amal $(\mathcal{N})$

In this section we consider the non-trivial members of Amal $(\mathcal{N})$. The following characterization theorem can be found in [9].

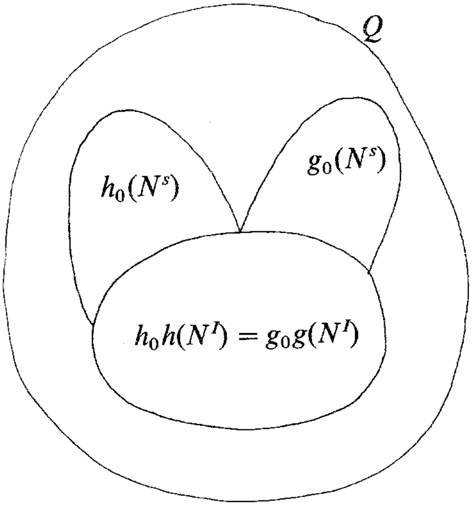


Figure 5

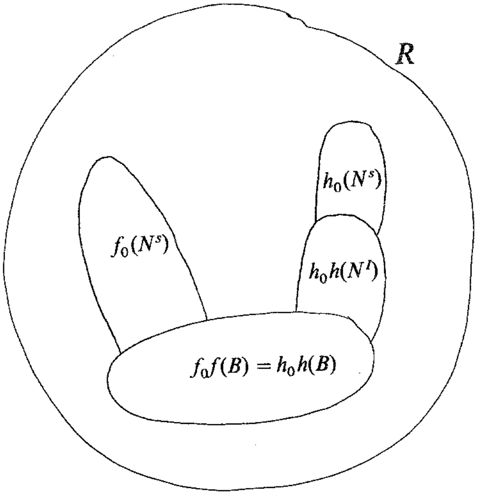


Figure 6

THEOREM 2.1. (1) (Bergman). *A lattice* $A \in \mathcal{N}$ *belongs to* Amal $(\mathcal{N})$ *iff* A *is a congruence extensile, subdirect power of* N *and for any subdirect representation* $f : A \to N^I$ *and any homomorphism* $g : A \to N$ *there is a homomorphism* $h : N^I \to N$ *such that* $g = hf$.

(2) *A finite lattice* $A \in \mathcal{N}$ *belongs to* Amal $(\mathcal{N})$ *iff* A *is a subdirect power of* N *and does not have* $\underline{3}$ *as a homomorphic image.*

Our next theorem gives a characterization of membership of Amal $(\mathcal{N})$ which is different from that in Theorem 2.1(1). This characterization is crucial for the proof of Theorem 3.1.

THEOREM 2.2. *For* $A \in \mathcal{N}$, *the following are equivalent*:

(1) $A \in$ Amal $(\mathcal{N})$.

(2) *If* $A \leq B \in \mathcal{N}$, *then every* $\underline{2}$*-congruence on* A *can be extended to a* $\underline{2}$*-congruence on* B *and* $\underline{3}$ *is not an image of* A.

(3) A *is a subdirect power of* N *and for any regular subdirect representation* $f : A \to N^I$ *and any homomorphism* $g : A \to N$ *there is homomorphism* $h : N^I \to N$ *such that* $g = hf$.

(4) A *is a subdirect power of* N, $\underline{3}$ *is not an image of* A *and if* $f : A \to N^I$ *is a regular subdirect representation then any* $\underline{2}$*-congruence on* A *can be extended to a* $\underline{2}$*-congruence on* N^I.

Proof. (1) $\Rightarrow$ (2). If $A \in \mathrm{Amal}(\mathcal{N})$ and $A \leq B \in \mathcal{N}$ then by Lemma 1.2.2 every $\underline{2}$-congruence on A can be extended to a $\underline{2}$-congruence on B. Further, Theorem 0.2.2(2) and Corollary 1.2.4 jointly imply that $\underline{3}$ is not an image of A.

(2) $\Rightarrow$ (1). If (2) holds then to show that (1) holds we again use Theorem 0.2.2(2). Let $g : A \to N$ be a homomorphism and $f : A \to B \in \mathcal{N}$ be an embedding. Since $\underline{3}$ is not an image of A we have that $g(A)$ must be isomorphic to one of the following lattices: $\underline{1}, N, \underline{2}$ or $\underline{2} \times \underline{2}$. For non-trivial cases: If $g(A) \cong N$ then the existence of h follows from Lemma 1.1.1. If $g(A) \cong \underline{2}$ of $g(A) \cong \underline{2} \times \underline{2}$ the existence of h follows from Lemma 1.2.1.

(1) $\Rightarrow$ (3) follows from Theorem 2.1(1).

Now by Corollary 1.2.4, (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (1). By Corollary 1.1.3 we have to show that if $g : A \to N$ is a homomorphism then there is $h : N^I \to N$ such that $hf = g$. By assumption the non-trivial images of A under g are $N, \underline{2}$ or $\underline{2} \times \underline{2}$, so that one can use arguments similar to those in the proof (2) $\Rightarrow$ (1).

COROLLARY 2.3. *If* $A \in \mathrm{Amal}(\mathcal{N})$ *then every finite distributive image of* A *is a Boolean algebra.*

Proof. The corollary follows from Theorem 2.2 since $\underline{3}$ is not an image of A.

We are now going to consider certain extensions of members of $\mathcal{N}$ which belong to $\mathrm{Amal}(\mathcal{N})$ (see Proposition 2.5).

We first need the following:

LEMMA 2.4. *Let* θ_0, θ_1 *be* $\underline{2}$-*congruences on* $B \in \mathcal{N}$ *with* $\underline{2}$-*sublattices* $\{u > v\}$ *and* $\{w > z\}$ *of* B *such that* $B/\theta_0 = \{u/\theta_0, v/\theta_0\}$ *and* $B/\theta_1 = \{w/\theta_1, z\theta_1\}$. *Assume the following*: *For some* $A \in \mathrm{Amal}(\mathcal{N})$ *and embeddings* $f_0, f_1 : A \to B$, *the restrictions* $\theta_0|_{f_0(A)}$ *and* $\theta_1|_{f_1(A)}$ *are distinct* $\underline{2}$-*congruences on* A. *Then* $\varphi = \theta_0 \cap \theta_1$ *is not a* $\underline{3}$-*congruence on* B.

Proof. Suppose that $B/\varphi \cong \underline{3}$. Since $\underline{3}$ is a projective lattice we may assume that $v = w$ so that $B/\varphi = \{u/\varphi, v/\varphi, z/\varphi\}$. Then for $k \in B$ there is $t \in \{u, v, z\}$ such that $(k, t) \in (\theta_0 \cap \theta_1)$. Since $\theta_0|_{f_0(A)}$ and $\theta_1|_{f_1(A)}$ are distinct $\underline{2}$-congruences on A without loss of generality we may assume that $\{u, v\} \subset f_0(A)$ and $\{v, z\} \subset f_1(A)$. Let $D \in \mathcal{N}$ be an amalgam of the diagram (A, f_0, B, f_1, B). By Theorem 2.2(2) θ_0, θ_1 can be extended to $\underline{2}$-congruences $\bar{\theta}_0, \bar{\theta}_1$ on D. Let $\bar{\varphi} = \bar{\theta}_0 \cap \bar{\theta}_1$. Then $\bar{\varphi}$ restricted on $f_0(A) = f_1(A)$ is a $\underline{3}$-congruence on A which contradicts Theorem 2.2.

As a corollary we have the following:

PROPOSITION 2.5. *Let* $B \in \mathcal{N}$, *and assume that, for any distinct* $\underline{2}$*-congruences* θ_0 *and* θ_1 *on* B *there is* $A \in \mathrm{Amal}\,(\mathcal{N})$ *and embeddings* $f_0, f_1 : A \to B$ *such that* $\theta_0|_{f_0(A)}$ *and* $\theta_1|_{f_1(A)}$ *are distinct* $\underline{2}$*-congruences on* A. *Then* $B \in \mathrm{Amal}\,(\mathcal{N})$.

Proof. It follows from the given conditions that if $B \leq C \in \mathcal{N}$ then every $\underline{2}$-congruence on B can be extended to a $\underline{2}$-congruence on C. Further from Lemma 2.4, B does not have $\underline{3}$-congruences. The proposition now follows from Theorem 2.2.

3. The main theorem

The principal result of this paper is the following:

THEOREM 3.1. $\mathrm{Amal}\,(\mathcal{N})$ *is an elementary class. It is closed under reduced products and therefore is determined by Horn sentences. Furthermore, if* B *is an image of* $A \in \mathrm{Amal}\,(\mathcal{N})$ *and* B *is a subdirect power of the pentagon then* $B \in \mathrm{Amal}\,(\mathcal{N})$.

Proof. We first prove the last statement of the theorem. Let B be a subdirect power of N and assume that B is an image of $A \in \mathrm{Amal}\,(\mathcal{N})$. It follows from Proposition 1.2.6 that if $B \leq N^I$ is any regular subdirect representation then any $\underline{2}$-congruence on B can be extended to a $\underline{2}$-congruence on N^I. Since $A \in \mathrm{Amal}\,(\mathcal{N})$, the lattice $\underline{3}$ is not an image of A and therefore $\underline{3}$ cannot be an image of B. Thus $B \in \mathrm{Amal}\,(\mathcal{N})$ by Theorem 2.2(2).

Next we show that $\mathrm{Amal}\,(\mathcal{N})$ is closed under direct products. Let $A = \prod_{\gamma \in \alpha} A_\gamma$ be a direct product of members of $\mathrm{Amal}\,(\mathcal{N})$. Without loss of generality we may assume that each A_γ is non-trivial. We use Proposition 2.5, that is we prove that $A \in \mathrm{Amal}\,(\mathcal{N})$ by showing that for any distinct $\underline{2}$-congruences θ_0, θ_1 on A there is $\gamma \in \alpha$ and embeddings $f_0, f_1 : A_\gamma \to A$ such that $\theta_0|_{f_0(A_\gamma)}$ and $\theta_1|_{f_1(A_\gamma)}$ are distinct $\underline{2}$-congruences on A_γ. Now if θ_0 and θ_1 are distinct $\underline{2}$-congruences on A then $A/(\theta_0 \cap \theta_1)$ is isomorphic either to $\underline{3}$ or $\underline{2} \times \underline{2}$. In either case we have $u, v, z \in A$ with $u > v > z$ such that

$$(u, v) \notin \theta_1, (v, z) \in \theta_1 \quad \text{and} \quad (v, z) \notin \theta_0, (u, v) \in \theta_0.$$

By Jónsson's Lemma there are filtral congruences φ_0, φ_1 on A such that $\varphi_0 \subset \theta_0$ and $\varphi_1 \subset \theta_1$. Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be ultrafilters which induce φ_0 and φ_1. We have

$$R = \{\beta \in \alpha : u_\beta > v_\beta\} \in \mathcal{D}_0$$

$$S = \{\beta \in \alpha : v_\beta > z_\beta\} \in \mathcal{D}_1.$$

There are threee possible cases:

(1) For some $\gamma \in \alpha$ the set $\{\gamma\}$ belongs to both $\mathscr{D}_0$ and $\mathscr{D}_1$.

(2) For each $\gamma \in \alpha$ the set $\{\gamma\}$ belongs to neither $\mathscr{D}_0$ nor $\mathscr{D}_1$.

(3) There is $\gamma \in \alpha$ such that $\{\gamma\}$ belongs to one ultrafilter and not the other.

If (1) holds then we can choose u, v, z so that $R = S = \{\gamma\}$ for some $\gamma \in \alpha$. For $\beta \in \alpha$ with $\beta \neq \gamma$ let a_β be an arbitrary but fixed element of A_β. In this case we can have $f_0 = f_1$ so that for $i \in \{0, 1\}$ the embedding $f_i : A_\gamma \to A$ is defined as follows:

> For $x \in A_\gamma$ the γth coordinate of $f_i(x) \in A$ is x and if $\beta \in \alpha$ with $\beta \neq \gamma$ then the βth coordinate of $f_i(x)$ is a_β.

Suppose now that (2) holds. Pick any $\gamma \in \alpha$. We have $(R - \{\gamma\}) \in \mathscr{D}_0$ and $(S - \{\gamma\}) \in \mathscr{D}_1$. First observe that since A_γ is non-trivial it is a subdirect power of N (Theorem 2.2). Thus there are at least two distinct epimorphisms $k, t : A_\gamma \to \underline{2} = \{0, 1\}$. The embedding $f_0 : A_\gamma \to A$ is defined as follows:

> For $x \in A_\gamma$ the γth coordinate of $f_0(x) \in A$ is x. If $\beta \in \alpha$ with $\beta \in R - \{\gamma\}$ then the βth coordinate of $f_0(x)$ is u_β if $k(x) = 1$ and v_β if $k(x) = 0$. For $\beta \notin R \cup \{\gamma\}$ the βth coordinates of $f_0(x)$ is an arbitrary but fixed element of A_β.

The embedding $f_1 : A_\gamma \to A$ is defined as follows:

> For $x \in A_\gamma$ the γth coordinate of $f_1(x) \in A$ is x. If $\beta \in \alpha$ with $\beta \in (S - \{\gamma\})$ then the βth coordinate of f_1 is v_β if $t(x) = 1$ and it is z_β if $t(x) = 0$. For $\beta \notin S \cup \{\gamma\}$ the βth coordinate of $f_1(x)$ is an arbitrary but fixed element of A_β.

The case (3) is a combination of (1) and (2). For instance if $\{\gamma\} \in D_0$ and $\{\gamma\} \notin D_1$ then $S - \{\gamma\} \in \mathscr{D}_1$. Thus f_0 is as in case (1) and f_1 as in case (2).

Thus we have shown that every direct product of members of Amal $(\mathscr{N})$ belongs to Amal $(\mathscr{N})$. Now if B is a reduced product of members of Amal $(\mathscr{N})$ then B must be a subdirect power of N (see Lemma 0.1.9). On the other hand B is an image of a product A of members of Amal $(\mathscr{N})$. Since $A \in$ Amal $(\mathscr{N})$ it follows that $B \in$ Amal $(\mathscr{N})$. In particular every ultraproduct of members of Amal $(\mathscr{N})$ belongs to Amal $(\mathscr{N})$ so that Amal $(\mathscr{N})$ is an elementary class (see [12]). It is determined by Horn sentences since it is closed under reduced products (see [4]).

4. Examples of varieties whose amalgamation class contains Amal $(\mathscr{N})$ as a subclass

The following result gives examples of varieties as indicated in the title above. The varieties considered here are by no means the only ones satisfying this property.

Let $\mathcal{V}$ be a variety generated by two maximal subdirectly irreducible members M and K where M is a finite modular lattice and K is either N or a lattice from the set $X = \{L_1, L_2\} \cup \{L_{15}^n : n \in \omega\}$.

PROPOSITION 4.1. Amal $(\mathcal{N}) \subset$ Amal $(\mathcal{V})$. *In particular by taking* $M = \underline{1}$ *or* $M = \underline{2}$ *then* Amal $(\mathcal{N})$ *is a subclass of the amalgamation class of the variety generated by a lattice from the set* X.

For Lemmas 4.2–4.4 we assume that $L \in X$.

LEMMA 4.2. *Let* $A \in$ Amal $(\mathcal{N})$ *and let* $g : A \to K$ *be a homomorphism.*

(1) *If* $K = L$ *then* $g(A) \cong N$ *or* $g(A) \cong \underline{2}^m$ *for some* $m \in \omega$.

(2) *If* K *is a finite modular lattice then* $g(A) \cong \underline{2}^m$ *for some* $m \in \omega$.

Proof. If $g(A)$ is a non-distributive lattice of $L_i (i = 1, 2)$ then $g(A) \cong N$ for otherwise $f(A) = L_i$. If $g(A)$ is a non-distributive sublattice of L_{15}^n then $g(A) \cong N$ for otherwise $g(A)$ will have $\underline{3}$ as its image which is impossible by Theorem 2.2. Now if $g(A)$ is a distributive sublattice of L, so that $g(A) \cong \underline{2}^m$ by Corollary 2.3. (2) follows from Corollary 2.3 since $g(A)$ is a distributive sublattice of K.

The proof of the next lemma is trivial and left for the reader.

LEMMA 4.3. (1) *Let* P_1 *and* P_2 *be distinct pentagons in* L *such that* $b_1 \in P_1$ *and* $b_2 \in P_2$ *(see Figures* 2 *and* 3*). Then the map* $b_1 \to b_2$ *can be extended to an automorphism* $\alpha : L \to L$ *such that* $\alpha(P_1) = P_2$.

(2) *Let* $\{u > v\}$ *be a* $\underline{2}$*-subset of* L. *Then* $\{u, v\}$ *can be chosen so that it is a* $\underline{2}$*-subset of some pentagon* P *in* L. *Furthermore there is a co-retraction* $\beta : L \to \{u, v\}$ *so that* $\beta(L) = \beta(P)$.

LEMMA 4.4. *Let* $K = N$ *or* $K = L$ *and let* M *be a finite modular lattice. Assume that* $A \leq B = K^I \times M^J$ *where* $A \in$ Amal $(\mathcal{N})$.

(1) *If* θ *is an* N*-congruence on* A *then* θ *can be extended to an* L*-congruence on* B.

(2) *If* ψ *is a* $\underline{2}$*-congruence on* A *then* ψ *can be extended to a* $\underline{2}$*-congruence on* B.

Proof. (1) By Jónsson's Lemma there is a filtral congruence φ on B such that $\varphi|_A \subset \theta$. Furthermore if $\mathscr{D}$ is an ultrafilter on $I \cup J$ which induces φ then clearly $I \in \mathscr{D}$ so that φ is a K-congruence on B.

(2) It is sufficient to consider only $K = L$. Since A is a subdirect power of N there is an N-congruence θ on A with $\theta \subset \psi$ Let φ be as in (1) so that $\varphi|_A \subset \theta \subset \psi$.

Let $h : B \to L$ be a homomorphism such that $\varphi = \ker h$. Then $\ker (\beta h)$ where β is as in Lemma 4.3(2) is a $\underline{2}$-congruence which extends ψ.

Proof of Proposition 4.1. By assumption $\mathscr{V}$ has at most two maximal subdirectly irreducible members M and K where M is a modular lattice and K is either N or K is a lattice from the set X. Let $A \in \text{Amal}\,(\mathscr{N})$. Since every member of $\mathscr{V}$ is embeddable into $B = K^I \times M^J$ for some index sets I and J, it is sufficient to show, by Theorem 0.2.2(2), that if $f : A \to B$ is an embedding and g is a homomorphism from A into K or M then there is a homomorphism h from B into K or M such that $g = hf$.

As before it is sufficient to consider only $K = L$. By Lemma 4.4 we have either $g(A) \cong N$ or $g(A) \cong \underline{2}^m$ for some $m \in \omega$. Let $g(A) \cong N$ and let $p : B \to L$ be an epimorphism such that $\ker p$ is an extension of $\ker g$ (see Lemma 4.4(1)). Then h is either p or αp where $\alpha : L \to L$ is as in Lemma 4.3(1). Now assume that $g(A) \cong \underline{2}^m$. The case $m = 0$ is trivial and Lemma 4.4(2) takes care of the case $m = 1$. If $m > 1$ then we apply the case $m = 1$ times. Finally if $g : A \to M$ then $g(A) \cong \underline{2}^m$ so that in this case the proof is the same as for $g : A \to K$ with $g(A) \cong \underline{2}^m$.

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